A New Lower Bound for the Circumference of Tough Graphs

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Abstract – A set of the vertices \( A \subseteq V(G) \) and not empty is called an independent set if every vertex of it is not adjacent to any other. We denote \( NC_{2}(G) = \min\{|N(u) \cup N(v)| : d(u,v) = 2\} \), where \( d(u,v) \) is length of the shortest path joining \( u \) and \( v \) if \( G \) is not complete. Moreover, we define \( NC_{k}(G) = \min\{|N(u) \cup N(v)| : d(u,v) = k\} \) and \( \sigma_{k}(G) = \min\{|d(v_{1}) + d(v_{2}) + \cdots + d(v_{k})| \} \) where \( \{v_{1}, v_{2}, \ldots, v_{k}\} \) is an independent set of vertices in case there exists the independent set \( \{v_{1}, v_{2}, \ldots, v_{k}\} \). A graph \( G \) called to be 1-tough if for every nonempty set \( S \subseteq V(G) \), the number component of \( G - S \), denoted by \( \alpha(G - S) \), less than or equal to \(|S|\). In this paper, we consider 1-tough graph \( G \) with \( n \) vertices and \( \sigma_{3}(G) \geq n \). Denote by \( c(G) \) the circumference (the length of a longest cycle) of \( G \), Bauer, Fan and Veldman proved that \( c(G) \geq \min\{n, 2NC_{2}(G)\} \). Vu Dinh Hoa strengthened this result by showing that \( c(G) \geq \min\{n, 2NC_{2}(G) + 4\} \).

Keywords – 1-tough graphs, dominating cycles, longest cycle, circumference.

I. INTRODUCTION

In this paper, we only consider finite undirected graph \( G \) with \( n \geq 3 \) vertices. We use the notation and terms in [3] and denote vertex and edge set of it by \( V(G) \) and \( E(G) \), respectively.

For any vertex \( v \in V(G) \), let \( N(v) \) and \( d(v) = |N(v)| \) denote the set of neighbors and the degree of \( v \), respectively. Moreover, for any subset \( X \subseteq V(G) \) we define \( N(X) = \bigcup_{v \in X} N(v) - X \). A set of vertices \( A \subseteq V(G) \) is called an independent set if every vertex of it is not adjacent to any other. We use \( \alpha \) to denote the cardinality of a maximum independent set of vertices of \( G \). A cycle \( C \) of \( G \) is called a dominating cycle, or briefly \( D \)-cycle, if \( V(C) \) is an independent set of vertices in \( G \). We denote \( NC(G) = \min\{|N(u) \cup N(v)| : uv \notin E(G)\} \) and \( NC_{2}(G) = \min\{|N(u) \cup N(v)| : d(u,v) = 2\} \) and \( NC_{k}(G) = \min\{|N(u) \cup N(v)| : d(u,v) = k\} \) where \( d(u,v) \) is length of the shortest path joining \( u \) and \( v \) if \( G \) is not complete. We set \( NC_{2}(G) = NC_{2}(G) = n - 1 \) if \( G \) is complete. Moreover, we define \( \sigma_{k}(G) = \min\{|d(v_{1}) + d(v_{2}) + \cdots + d(v_{k})| \} \) where \( \{v_{1}, v_{2}, \ldots, v_{k}\} \) is an independent set of vertices in case there exists the independent set \( \{v_{1}, \ldots, v_{k}\} \), otherwise \( NC_{k}(G) = n - \alpha \) and \( \sigma_{k}(G) = k(n - \alpha) \). Sometimes we write \( N(a,b) \) instead of \( N(a) \cup N(b) \).

A graph \( G \) is called to be 1-tough if for every nonempty set \( S \subseteq V(G) \), the number component of \( G - S \), denoted by \( \alpha(G - S) \), less than or equal to \(|S|\). Two disjoint paths of \( G \) is said to be to adjacent if there is at least one vertex of a path adjacent to an vertex of the other. Denote by \( c(G) \) the circumference (the length of a longest cycle) of \( G \), the following bound on \( c(G) \) due to Bauer, Fan and Veldman

**Theorem 1 (Theorem 26 in [1]).** If \( G \) is 1-tough and \( \alpha_{3} \geq n \), then \( c(G) \geq \min\{n, 2NC_{2}(G)\} \).

Vu Dinh Hoa strengthened this result in [4].

**Theorem 2 (Theorem 2 in [4]).** If \( G \) is a 1-tough graph with \( \alpha_{3} \geq n \geq 3 \), then \( c(G) \geq \min\{n, 2NC_{2} - n + 5\} \).

Bauer, Veldman and Fan also conjectured:

**Conjecture 1 (Conjecture 27 in [1]).** If \( G \) is 1-tough and \( \alpha_{3} \geq n \), then \( c(G) \geq \min\{n, 2NC_{2} + 4\} \).

The conjecture remains unsolved even some people have tried to prove it (see [9]...). The proof of Tri Lai [9] is not correct, for example the proof for the **Proposition 26**, and the definition of bad path is not strong enough. In the mean time, a lot of new bounds for the length of longest cycle was found. For examples:

**Theorem 3 (Theorem 3 in [5]).** Let \( G \) be a graph of ordeh\( n \). If \( G \) is 3-connected then \( c(G) \geq \min\{n, \frac{3}{2}(n+1)\} \); if \( G \) is 4-connected, \( c(G) \geq \min\{n, 2NC\} \). Furthermore, these bounds are both sharp.

**Theorem 4 (Theorem 2 in [8]).** Let \( G \) is 3-connected\( K_{1,3}\)-free graph of ordeh\( n \) such that \( NC_{2} \geq (2n - 6)/3 \) then \( G \) is hamiltonian.

**Theorem 5 (Theorem 4 in [6]).** Let \( G \) be a 2-connected graph such that \( \max\{|N(u), d(v)| \geq c/2 \) for each pair of nonadjacent vertices \( u \) and \( v \) in an induced claw, and \(|N(X) \cap N(Y)| \geq 2 \) for each pair of nonadjacent vertices \( X \) and \( Y \) in an induced modified claw. Then \( G \) contains either Hamiltonian cycle or a cycle of length at least.

Our goal in this paper is to prove the conjecture posed by Bauer, Veldman and Fan in [1].

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II. PRELIMINARIES

For what follows, we assume that $G$ is $t$-tough with $t \geq n$ and nonhamiltonian. For a longest cycle $C$, we denote $by(C) = \max\{d(u) : u \not\in C\}$ and $\mu(G) = \max\{\mu(C) : l(C) = c(G)\}$ where $l(C)$ is the length of $C$. We consider a longest cycle $C$ and a vertex $v \in C$ such that $\mu(C) = \mu(G)$ and $d(v) = \mu(C)$. On $\overline{C}$ (with a given orientation), we denote the predecessor and successor of $x \in C$ (along the direction of $C$) by $x^-$ and $x^+$, respectively, and $x^- = (x^+)^-$. In general, $x^i = (x^{i-1})^+$ and $x^{-i} = (x^{-i-1})^-$. Let $S_N = N(v)^+ \cap N(v)^-$. The arc joining two vertexes $x$ and $y$ of $C$, along the direction of $C$, is denoted by $xy$. Similarly, the arc joining two vertexes of $C$, along the direction of $\overline{C}$, is denoted by $xy$. Moreover, for any $A \subseteq V(G)$, we write $A^+ = \{x^+ : x \in A\}$ and $A^- = \{x^- : x \in A\}$. We begin with the following lemmas:

**Lemma 1 (Theorem 5 in [2]).** Every longest cycle $C$ is a $D$-cycle.

**Lemma 2 (Proof of Theorem 9 in [2]).** $d(v) \geq \frac{\sigma_t}{3}$.

**Lemma 3 (Lemma 9 in [4]).** $|S| \geq \sigma_t - n + 4$.

For what follows we use a lemma of Woodall, sometimes called Hopping Lemma, in [7]:

**Lemma 4.** Let $C'$ be a cycle of length in a graph $G$. Assume that $G$ contains no cycle of length $+1$ and no cycle $C'$ of length with $d(C') < d(G)$ and $v$ is an isolated vertex of $G - C$. Set $V_0 = \emptyset$ and for $t > 1$:

$$X_{i+1} = N(Y_i \cup \{v\}),$$

$$Y_{i+1} = (X_i \cap V(C))^+ \cap (X_i \cap V(C))^-. $$

Set $X = \bigcup_{i=1}^{\infty} X_i$ and $Y = \bigcup_{i=1}^{\infty} Y_i$. Then

(a) $X \subseteq V(C)$,

(b) $|fx_i, x_2 \in X$, then $x_1^+ \neq x_2$.

(c) $X \cap Y = \emptyset$.

By (a), (b) and (c) of Lemma 4, respectively, we easily conclude that:

**Corollary 1.** Assume that $G, C, X, Y$ satisfy the hypothesis of Lemma 4, then:

(a) There are no edges joining a vertex of $Y$ with a vertex of $G - C$.

(b) $X \cap X^+ = \emptyset$, and therefore $|C| \geq 2|X|$.

(c) $Y$ is an independent set of vertices.

Assume that $X, Y$ are the vertex sets mentioned in the assumption of Lemma 4. First, we fixed a vertex $x_0 \in S$ and denote the vertices of $X$ by $x_1, x_2, \ldots, x_m (m = |X|)$, occurring consecutively on $C$ such that $x_i = x_i^+$ and $x_m = x_0^+$. For $1 \leq i \leq m - 1$, we write $x_i = x_i^+$ and $x_i = x_i^{i+1}$ by Lemma 4. $x_1^+ \neq x_1$ and therefore $C - X$ is in fact the union of $|X|$ disjoint paths $\overline{C} w_i (i = 1, m)$ on $C$, calling $arc$es. An arc with $p$ vertices is called a $p$-arc. The $p$-arc $w_i$ is an 1-arc if and only if $u_i \equiv w_i \in Y$. The next lemma can be proved using Corollary 1 and Lemma 1.

![Figure 1](image.png)  

**Lemma 5.**

(a) $Y \cup V(G - C)$ is an independent set of vertices.

(b) $|l(C) - 2|X| + 2$.

Proof. 

(a) By (a) of Corollary 1, there is no edge joining a vertex of $Y$ with a vertex of $V(G - C)$. By (c) of Corollary 1, $Y$ is an independent set of vertices. By Lemma 1, $V(G - C)$ is an independent set of vertices. Thus, $Y \cup V(G - C)$ is an independent set of vertices.

(b) By (b) of Corollary 1, $l(C) \geq 2|X|$. If $l(C) \leq 2|X| + 1$ then $C - X$ is an union of $|X|$ 1-arcs or an union of one 2-arc and $(|X| - 1)$ 1-arcs. The 1-arcs of $C - X$ are in fact the vertices of $Y$. By (a), the graph $G - X$ contains at least $|X| + 1$ components, a contradiction to the toughness of $G$.

By the toughness of $G$, by $X = N(Y)$ and by (a) of Lemma 5, we easily conclude that:

**Corollary 2.** $C - X$ contains at least two arcs with length $\geq 2$.

By $d(v, s) = 2$ for any $s \in S$ and by $N(s) \cup N(v) \subseteq X$, $NC2 \leq |N(s) \cup N(v)| \leq |X|$. If $NC2 \neq |X|$ then by Lemma 5, $c(G) \geq 2|X| + 2 \geq 2NC2 + 4$ and Conjecture 1 is proved. For what follows we assume that $NC2 = |X|$ and $c(G) \leq 2|X| - 3$ and will show that it leads to a contradiction. By $NC2 = |X|$, $X = N(s) \cup N(v)$ for any $s \in S$ and we conclude from Lemma 4 in [4] that:

**Lemma 6.** $V(G - C) \cup X^+$ is an independent set. Similarly, $V(G - C) \cup X^-$ is an independent set.
The following definition will help us to get shorter proofs.

**Definition 1.** A path $P$ in $G$ is called a bad-path if it has one of the following two forms:
1. $P$ consists of all vertices $vx_iw_{j-i-1}$, and the ends of $P$ are $x_k \in N(v) \cap N(s_0)$ and $x_i \in X$ for $1 \leq i < m$ and $i \neq k$.
2. $P$ consists of all vertices in $x_mw_iu_{i+1}$, and the ends of $P$ are $x_k \in N(v) \cap N(s_0)$ and $x_i \in X$ for $1 \leq i < j < m$ and $j \neq k$.

**Lemma 7.** There are no bad-paths in $G$.

**Proof.** Assume the contrary that $G$ has a bad-path $P$. Without loss of generality, we assume that $P$ is a bad-path of form 1. We will show a cycle $C'$ is longer than $C$, then get a contradiction. As remark before that $X = N(x) \cup N(u)$ for any $x \in S$ and specially $X = N(s_0) \cup N(v)$. Therefore, there are only four possible cases as follows. In each case we will get a contradiction by constructing a cycle $C'$ which is longer than $C$.

Case 1: $x_i \in N(s_0)$, then $C = (x_kP)x_iS_jX_mw_1x_k$.

Case 2: $x_i \notin N(s_0)$, then $C = (x_kP)x_iw_1x_jX_mw_1x_k$.

Case 3: $x_i \notin N(s_0)$, then $C = (x_kP)x_iw_1x_jX_mw_1x_k$.

Case 4: $x_i \notin N(s_0)$, then $C = (x_kP)x_iw_1x_jX_mw_1x_k$.

**Lemma 8.** If $w_iu_j \in E(G)$ for $1 \leq i < j < m$, then $x_iw_{i+1}, x_j \notin N(v) \cap N(s_0)$ and $x_iw_{i+1}, x_j \notin N(v) \cap N(s_0)$.

**Proof.** Assume the contrary that $x_iw_{i+1}, x_j \in N(v) \cap N(s_0)$. Then $P = (x_kP)x_iw_{i+1}x_jw_1x_k$ is a bad-path, which contradicts to Lemma 7. Thus, we have $x_iw_{i+1}, x_j \notin N(s_0)$. Similarly, by reversing the direction of $C$, we have $x_iw_{i+1}, x_j \notin N(w_{m-1})$.

**Lemma 10.** Assume that $u_iw_j \in E(G)$ for some $1 \leq i < j < m$. Then $x_iw_{i+1}, x_j \notin N(u_k)$ for every $i < k < j$.

**Proof.** Assume that $z$ is vertex in $u_iw_j$ such that $zu_i \in E(G)$ and $z \notin N(x_1)$. Similarly, if $zu_i \in E(G)$ and $z \notin N(x_m)$.

**Lemma 12.** Assume that $u_iw_j \in E(G)$ for some $1 \leq i < j < m$. If there exists a vertex $x_k \in u_iw_j$ such that $x_k \notin N(s_0)$ and $N(s_0) \cap u_iw_j \notin N(u_k)$.

**Proof.** Let $x_k$ be a vertex in $N(u_v) \cap N(s_0) \cap u_iw_j$. If $x_k \notin N(s_0) \cap u_iw_j$ then $P = (x_kP)x_iw_{i+1}x_jw_1x_k$ is a bad-path, which contradicts to Lemma 7. If $x_k \notin N(s_0) \cap u_iw_j$ then $P = (x_kP)x_iw_{i+1}x_jw_1x_k$ is a bad-path, a contradiction. Thus, we have $x_iw_{i+1}, x_j \notin N(u_k)$.

**Lemma 13.** Assume that $C$ has a $p$-arc $u_iw_1$ and a $q$-arc $u_iw_j$ within $(p, q)$, $i \neq j$. Then $u_iw_j \notin E(G)$.

**Proof.** Assume to the contrary that $u_iw_j \in E(G)$. Without loss of generality, we assume that $p = 2$, then we have $P = (x_kP)x_iw_{i+1}x_jw_1x_k$ is a bad-path, which contradicts to Lemma 7.

**Lemma 14.** Assume that $C$ has a $p$-arc $u_iw_1$ and a $q$-arc $u_iw_j$ within $(p, q)$, $\geq 2$, $i < j$ such that $u_iw_j \in E(G)$ and there are no $k$-arcs $(k \geq 2)$ on $x_iw_{j-i+1}$. If $N(w_{i+1}) \subseteq X \cup \{w_j\}$ or $N(u_i) \subseteq X \cup \{u_j\}$, then $x_iw_{i+1} \equiv x_j$.

**Proof.** Assume to the contrary that $x_iw_{i+1} \equiv x_j$.

**Lemma 15.** Assume that $C$ has a $p$-arc $u_iw_1$ and a $q$-arc $u_iw_j$ within $(p, q)$, $\geq 2$, $i < j$ such that there are no $k$-arcs $(k \geq 2)$ on $x_iw_{j-i+1}$. If $N(w_{i+1}) \subseteq X \cup \{u_j\}$ or $N(u_i) \subseteq X \cup \{w_j\}$, then $u_iw_j \notin E(G)$.

**Proof.** Assume to the contrary that $u_iw_j \in E(G)$. Without loss of generality, we assume that $N(w_j) \subseteq X \cup \{w_j\}$. By Lemma 8, we have $x_iw_{i+1}, x_j \notin N(u_v) \cap N(s_0) \cap u_iw_j$.

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N(y). Clearly, d(y, u_i) = d(y, w_{m-1}) = 2. so |N(y, u_i)|, |N(y, w_{m-1})| ≥ NC2(G) = |X|. If x_i ≠ x_j, then u_j ∈ Y. By Lemma 9, we have x_{i+1}, x_j ∈ N(u_i), and therefore N(y, u_i) ⊆ X − {x_i+1, x_j}. A contradiction. Thus x_i ≡ x_j. Similarly, we have x_{i+1} ≡ x_m and w_j ≡ w_{m-1}. By Lemma 9, we have x_{i+1}, x_j ∈ N(w_{m-1}), so N(y, w_{m-1}) ⊆ X ∪ {w_{m-1} − {x_i+1, x_j}}. If x_{i+1} ≠ x_j, then |N(y, w_{m-1})| ≤ |X ∪ {w_{m-1} − {x_i+1, x_j}}| ≤ |X| − 1, a contradiction. Thus x_{i+1} ≡ x_j, and therefore |S| = 1, which contradicts to Lemma 3 that |S| ≥ σ_3 − n + 4 ≥ 4.

By (c) g ≤ 2|X| + 3 and by (b) of Lemma 5, either g = 2|X| + 3 or g = 2|X| + 2. We consider two cases:
Case 1. c(G) = 2|X| + 2. In this case, by Corollary 2, C − X contains two 2-arc and some 1-arcs.
Case 2. c(G) = 2|X| + 3. In this case, by Corollary 2, there are only two possible cases:
Case 2.1. C − X contains one 2-arc, one 3-arc and some 1-arcs.
Case 2.2. C − X contains three 2-arc and some 1-arcs.

III. PROOF FOR THE CASES

A. Case 1. c(G) = 2|X| + 2. C − X contains two 2-arc and some 1-arcs.
Assume that the two 2-arcs are u_i w_i and u_j w_j with 1 ≤ i < j < m. We have the following Claims (from Claim 1 to Claim 4):

Claim 1. u_i w_i ∈ E(G) and u_j w_j ∈ E(G).
Proof. Assume that u_i w_i ∈ E(G), then u_i w_i ∈ E(G), otherwise, by Lemma 6, ω(G) − X ≥ |X| + 1, which contradicts the toughness of G. By Lemma 6, N(u_i) ⊆ X ∪ {u_i}, and by Lemma 15, u_i w_j ∈ E(G), a contradiction.
Thus, u_i w_i ∈ E(G), and by Lemma 13, u_i w_j ∈ E(G).
By Lemma 6, Lemma 14 and Claim 1, we get:
Claim 2. x_{i+1} ≡ x_j.
Note: by Claim 1 and Claim 2, d(w_i, u_j) = 2. By Lemma 10, we have x_i, x_{i+1} ∈ N(w_i, u_j).

Claim 3. x_j ∈ N(u) ∩ N(s_3).
Proof. Assume the contrary that x_j ∈ N(u) ∩ N(s_3). By Lemma 12, x_{i+1} x_m ∈ N(w_j, u_j). By Lemma 6, N(w_j, u_j) ⊆ X ∪ {u_i, w_j} − {x_i, x_m, x_{i+1}, x_j}. By |N(w_j, u_j)| = |N(w_j) ∪ N(u_j)| ≥ NC2(G) = |X|, we have x_i ≡ x_1 and x_{i+1} ≡ x_m.
Thus, |S| = 1, which contradicts to Lemma 3 that |S| ≥ σ_3 − n + 4 ≥ 4.

Claim 4. x_{i+1} ≡ x_j.
Proof. Assume that x_i ≠ x_1, then u_i ∈ Y. By Claim 3, there exist y ∈ {v, s_3} such that y ∉ E(G). Clearly, d(u_i, y) = 2. If u_i y ∈ E(G), then N(u_i, y) ⊆ X − {x_1}, and therefore |N(u_i, y)| ≤ |X| − 1, which contradicts |N(u_i, y)| ≥ NC2(G) = |X|. Thus, u_i y ∉ E(G).
By Lemma 11, x_i = N(w_i, u_j), and by Lemma 6, N(w_j, u_j) ⊆ X ∪ {u_i, w_j} − {x_1, x_m, x_{i+1}}. Therefore, |N(w_i, u_j)| = |X| − 1, which contradicts d(w_i, u_j) = 2 as noted above.
Thus, x_i ≡ x_1. Similarly, we have x_{i+1} ≡ x_m.
By Claim 2 and Claim 4, |S| = 1, which contradicts Lemma 3. So, the Case 1 does not happen.

B. Case 2. c(G) = 2|X| + 3. In this case, there are only two possible cases:
Case 2.1. C − X contains one 2-arc, one 3-arc and some 1-arcs.
Assume that the 3-arc is u_i u_j w_j and the 2-arc is u_i w_j. Without loss of generality, we assume that i < j. We have two following Propositions:

Proposition 1. u_i w_i ∈ E(G).
Proof. Assume to the contrary that u_i w_i ∈ E(G). We have the following Claims (from Claim 5 to Claim 10):
Claim 5. u_i u_j w_j ∈ E(G).
Proof. If u_i u_j w_j ∈ E(G) then P = (x_m C u_i u_j w_j C x_j) is a bad-path, which contradicts Lemma 7. If u_i w_j ∈ E(G) then P = (x_m C u_i u_j w_j C x_{i+1}) is a bad-path, which contradicts Lemma 7.

Claim 6. u_j w_i ∈ E(G) and u_i w_j ∈ E(G). Moreover, x_{i+1} ≡ x_j.
Proof. If u_j w_i ∈ E(G), then by Lemma 6 and by Claim 5, ω(G) − X ≥ |X| + 1, so G is not 1-tough, a contradiction.
If u_i w_j ∈ E(G), then by Lemma 13, u_i w_j ∈ E(G). Therefore, by Lemma 6 and by Claim 5, N(w_j) ⊆ X ∪ {u_j}. By Lemma 15, we have u_j w_i ∈ E(G), a contradiction.
Thus, u_i w_j ∈ E(G). By the toughness of G, by Claim 5 and by Lemma 6, u_i w_j ∈ E(G). By u_j w_i ∈ E(G) and by Lemma 6, N(u_j) ⊆ X ∪ {u_j}. By Lemma 14, x_{i+1} ≡ x_j.

Claim 7. d(w_i, u_j) = 2 and x_{i+1}, x_j ∈ N(w_i, u_j).
Proof. By Claim 6, d(w_i, u_j) = 2. By Lemma 10, x_j, x_{i+1} ∈ N(w_i, u_j).

Claim 8. x_i ≡ N(v) ∩ N(s_3).
Proof. Assume to the contrary that x_i ∈ N(v) ∩ N(s_3). Since Lemma 12 we have x_i x_m ∈ N(w_i, u_j). By Lemma 6 and by Claim 7, N(w_i, u_j) ⊆ X ∪ {u_j, u_i, w_j} − (x_i, x_m, x_{i+1}). Because |N(w_i, u_j)| ≥ NC2(G) = |X|, so x_i ≡ x_{i+1} or x_{i+1} ≡ x_m if x_i ≡ x_{i+1}, then x_{i+1} ≡ x_m because of |S| ≥ 4, and there are at least three 1-arcs on x_{i+1} C x_m. Since N(w_i, u_j) ⊆
Proof. Assume the contrary that \( x_i \not= x_j \) and \( x_{i+1} \not= x_m \). By Claim 9, \( x_i, x_{i+1}, x_m \not\in M(w_i, u_j) \). By Lemma 5, \( u_j \not\in \mathcal{N}(w_i) \). Therefore, by Lemma 1, we have \( u_j \not\in \mathcal{N}(w_i) \). Similarly, we have \( u_j \not\in \mathcal{N}(w_m) \).

Claim 10. If \( x_i \not\equiv x_{i+1} \not\equiv x_m \) then we get a contradiction.

Proof. Assume the contrary that \( x_i \not= x_j \) and \( x_{i+1} \not= x_m \). By Claim 7, it is clear that \( x_j \not\in \mathcal{N}(w_i, u_j) \). We consider the case \( x_j \not= x_i \).

Assume that \( u_j \not\in \mathcal{N}(w_i) \). Clearly that \( d(v, u_j) = 2 \) and \( u_j \not\in \mathcal{N}(v, u_j) \). This implies that \( |\mathcal{N}(v, u_j)| \geq |\mathcal{N}(u_j, v)| \), a contradiction. Therefore, \( u_j \not\in \mathcal{N}(w_i) \).

Claim 9. \( x_i \not\equiv x_{i+1} \not\equiv x_m \) then \( u_j \not\in \mathcal{N}(w_i) \).

Proof. Assume the contrary that \( x_i \not= x_j \) and \( x_{i+1} \not= x_m \). By Claim 7, \( x_i, x_{i+1}, x_m \not\in M(w_i, u_j) \). By Lemma 5, \( \mathcal{N}(w_i) \subseteq \mathcal{N}(w_i) \) and \( \mathcal{N}(w_i, u_j) \subseteq \mathcal{N}(w_i) \). Therefore, \( u_j \not\in \mathcal{N}(w_i) \).

This finishes the proof of the Proposition 1.

Proposition 2. \( u_j, w_j \not\in \mathcal{N}(w_i) \) and \( u_j, w_j \not\in \mathcal{N}(w_i) \).

Proof. If \( u_j, w_j \not\in \mathcal{N}(w_i) \), then by Lemma 13, \( u_j, w_j \not\in \mathcal{N}(w_i) \). Therefore, by Lemma 6 and by Proposition 1, \( M(u_j) \not\subseteq \mathcal{N}(w_i) \). By Lemma 15, \( u_j, w_j \not\in \mathcal{N}(w_i) \).

Thus, \( u_j, w_j \not\in \mathcal{N}(w_i) \).

Claim 11. If \( u_j \not\in \mathcal{N}(w_i) \) then \( u_j \not\in \mathcal{N}(w_i) \) have the following claims (from Claim 11 to Claim 15):

Claim 11. \( u_j \not\in \mathcal{N}(w_i) \) then \( u_j \not\in \mathcal{N}(w_i) \) since \( u_j \not\in \mathcal{N}(w_i) \) is adjacent to \( u_k \).

Claim 11. \( u_j \not\in \mathcal{N}(w_i) \) then \( u_j \not\in \mathcal{N}(w_i) \).

Claim 12. \( u_j \not\in \mathcal{N}(w_i) \) then \( u_j \not\in \mathcal{N}(w_i) \).

Proof. If \( u_j \not\in \mathcal{N}(w_i) \) then \( u_j \not\in \mathcal{N}(w_i) \) and \( u_j \not\in \mathcal{N}(w_i) \). If \( u_j \not\in \mathcal{N}(w_i) \) then \( u_j \not\in \mathcal{N}(w_i) \).

Claim 12. \( u_j \not\in \mathcal{N}(w_i) \) then \( u_j \not\in \mathcal{N}(w_i) \).

Note: Now we consider the pair \( \{w_i, u_j\} \) with distance 2. By Lemma 7, we have \( \{w_i, u_j\} \not\in \mathcal{N}(w_i, u_j) \).

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Claim 14. $x_i, x_m \in \mathcal{N}(w_i, u_j)$.
Proof. First, we prove that $x_i \in \mathcal{M}(w_i, u_j)$. The proof for the case $x_m \in \mathcal{M}(w_i, u_j)$ is similar. If $x_i \equiv x_i$ then by Lemma 10, $x_i \in \mathcal{M}(w_i, u_j)$. For what follows, we assume that $x_i \not\equiv x_i$. Clearly, $u_j \in Y$.
If $x_i \in \mathcal{N}(v') \cap \mathcal{N}(s \setminus v')$, then by Lemma 12, $x_i \in \mathcal{N}(w_i, u_j)$. If $x_i \not\in \mathcal{N}(v') \cap \mathcal{N}(s \setminus v')$, let $y \in \{s, v'\}$ such that $x_i \not\in \mathcal{N}(y)$. Clearly, $d(y, u_i) = 2$. If $x_i, u_i \not\in \mathcal{E}(G)$, then $\mathcal{N}(y, u_i) \subseteq X - \{x_i\}$ and therefore $|\mathcal{N}(y, u_i)| \leq |X| - 1$, which contradicts to the fact that $|\mathcal{N}(y, u_i)| \geq MC(2) = |X|$. Thus, $u_i, x_i \in \mathcal{E}(G)$. By Lemma 11, we have $x_i \in \mathcal{N}(w_i, u_j)$.

Claim 15. $x_i \equiv x_j \iff x_{i+1} \equiv x_m$.
Proof. Assume to the contrary that $x_i \not\equiv x_j$ and $x_{i+1} \not\equiv x_m$. By Lemmas 6, 10 and by Claim 12, 14, we have $\mathcal{N}(w_i, u_j) \subseteq \mathcal{E}(X) \cup \{w_i, w_j, w_k\} - \{x_i, x_m, x_{i+1}, x_{i+2}\}$. By Claim 13, $d(w_i, u_j) = 2$, so $|X| - 2 \geq |\mathcal{N}(w_i, u_j)| \geq MC(2) = |X|$, a contradiction.

Assume without loss of generality that $x_i \equiv x_j$. By Claim 13 and by $|S| \geq 4$, $x_{i+1} \not\equiv x_m$ and there are at least three 1-arcs on $x_{i+1} \mathcal{E} X$. Since $|X| = MC(2) \leq |\mathcal{N}(w_i, u_j)| \leq |X| \cup \{w_i, w_j, w_k\} = \{x_i, x_m, x_{i+1}, x_{i+2}\}$, so $x_i \in \mathcal{N}(w_i, u_j)$ for every $k \leq l < m$, and we will show that $u_i, x_{i+1} \not\in \mathcal{E}(G)$, so $\mathcal{N}(w_i, u_j) = \mathcal{E}(G)$. Indeed, assume otherwise that $u_i, x_{i+1} \in \mathcal{E}(G)$. We have two cases:

(1) If $x_i \not\in \mathcal{E}(G)$ then $P = (x_m \mathcal{E} u_i \mathcal{E} u_j \mathcal{E} x_j)$ is a bad-path, a contradiction.
(2) If $x_i \not\in \mathcal{E}(G)$ then $P = (x_m \mathcal{E} u_i \mathcal{E} u_{i+1} \mathcal{E} x_i)$ is a bad-path, a contradiction.

Clearly, $u_{i+2}, u_{i+3} \in Y$ and $\mathcal{N}(u_{i+2}, u_{i+3}) \subseteq X - \{x_{i+1}\}$ and $d(u_{i+2}, u_{i+3}) = 2$, so we conclude $|X| = MC(2) \leq |\mathcal{N}(u_{i+2}, u_{i+3})| \leq |X| - 1$, a contradiction. Thus, $u_i, x_i \not\in \mathcal{E}(G)$. By Lemma 11, $|X| = MC(2) \leq |\mathcal{N}(w_i, u_j)| \leq |X|$. By Lemma 11, we have $x_{i+1} \equiv x_m$.
Moreover, \( w_{t-1} \in I \) and \( w_{t-1} \subseteq E(G) \). Indeed, assume that \( w_{t-1} \subseteq E(G) \). Then:

1. If \( x_i \in N(w_i) \) then \( P = (x_i \perp w_{t-1} \perp x_i \perp w_{t-1} \perp x_i) \) is a bad-path, a contradiction.
2. If \( x_i \in N(u_j) \) then \( P = (x_i \perp w_{t-1} \perp x_i \perp w_{t-1} \perp x_i) \) is a bad-path, a contradiction.

Therefore, \( M(w_{t-1}, w_{t-1}) \subseteq X - \{x_i\} \) and \( |M(w_{t-1}, w_{t-1})| \leq |X| - 1 \). Because \( d(w_{t-1}, w_{t-1}) = 2 \), this implies that \( |M(w_{t-1}, w_{t-1})| \geq M2(G) = |X| \), a contradiction.

Thus, \( x_i \equiv x_j \). Similarly, we have \( x_{t+1} \equiv x_m \) and there are at least three \( x_{t+1} \)-arcs on \( x_{t+1} \perp x_k \).

If \( u_j, w_k \in E(G) \) then by Lemma 13, \( u_j, w_k \in E(G) \). By Lemma 6 and by Proposition 3, \( M(u_j, w_k) \subseteq X \cup \{w_k\} \). By Lemma 14, \( x_{t+1} \equiv x_k \), a contradiction. Therefore \( u_j, w_k \in E(G) \). By Lemma 6 and by Proposition 3 and by Claim 17, we have \( M(u_j, w_k) \subseteq X \cup \{u_j, w_k\} \).

Arguing similarly to the proofs of Claim 18, we have \( u_j \in Y \) and \( u_j, x_{j+1} \notin E(G) \) for every \( j + 1 < t \). Therefore \( M(u_j, w_k) \subseteq X - \{x_{j+1}\} \) and \( |M(u_j, w_k)| \leq |X| - 1 \), which contradicts to the fact that \( |M(u_j, w_k)| \geq M2(G) = |X| \). This finishes the proof of the Proposition 4.

Finally, we conclude that \( u_j, w_k \), \( u_j, w_j, u_k, w_k \) are pairwise non-adjacent arcs. By Lemma 6, \( \omega(G - X) \geq |X| + 1 \), which contradicts to the assumption that \( G \) is 1-tough.

Thus, the Case 2.2. does not happen.

IV. CONCLUSIONS

In this paper, we present some lower bounds for the circumference of tough graphs and prove the conjecture posed by Bauer, Fan and Veldman [1], namely, that: “If \( G \) is 1-tough graph with \( \sigma \geq n \), then \( c(G) \geq m \) in \( n, 2M2(G) + 4 \).”

REFERENCES