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A Probabilistic Interval Division Method for Solving Nonlinear Equations

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Abstract - In this paper a new iterative method is presented for finding the zeros of a non linear equation using the concept of probability. In this method a sub bracket is produced after bracketing the root by dividing the initial interval into a number of subintervals of equal length. In this work, after assigning a probability to each subinterval, sub bracketing is done by not evaluating the functional value at the each end points of every subinterval but at the end points of some subintervals whose number is effectively less than the total number of subintervals. The method is validated by several numerical examples.

Keywords - Nonlinear equation, Interval division, Probability.

I. INTRODUCTION

Let us consider the non linear equation f(x) = 0. Let $x^* \in [a_0, b_0]$ be a simple root of f(x) = 0 and $f \in C^2[a_0, b_0]$, where $b_0 - a_0 > 0$. Many literature appeared in past for the solution of a non linear equation and most of researchers gave their concentration through deterministic approach [1 - 6]. The concept of probability was introduced by some authors to solve quasi linear and linear parabolic equations and probabilistic approach was used to solve those parabolic equations numerically [7-10]. But sofar, probabilistic concepts are not used by any researcher to find the solution of non linear equation. This work may be the first one for solving an equation of the form f(x) = 0 by using probability. It has been shown that by assigning suitable probability in suitable interval the equation can also be solved.

Let us divide the initial interval $[a_0, b_0]$ into k subintervals of equal length. The real root x^* obviously lies in any one of these k subintervals. The main aim of this paper is to find out that particular subinterval in which the probability of lying the real root x^* is maximum. This is done by assigning a probability to each subinterval and the subinterval having the highest probability has the chance of containing the root x^*.

Since x^* lies between a_0 and b_0 therefore $f(a_0)$ and $f(b_0)$ are of opposite in sign i.e. $f(a_0).f(b_0) < 0$. It is being observed that the ratio $\frac{f(b_0)}{f(a_0)}$ play an important, significant role and is the key factor to locate the root x^* into the particular subinterval. We write $\lambda = \frac{f(b_0)}{f(a_0)}$. Further, it is being observed that another important factor to locate the root x^* into the particular subinterval is $\beta = \frac{1}{1-\lambda}$. We call this as β-value of the non linear equation f(x) = 0 which actually specify the location of the root x^*.

II. THE METHOD

Let us divide the whole interval into k subintervals of equal length $\frac{b_0 - a_0}{k}$. Certainly the root x^* lies in any one of these subintervals. By assigning a probability to each subinterval, a particular subinterval is chosen which has a maximum probability of containing the root x^*. Let $I_1, I_2, I_3, \ldots \ldots, I_k$ are the k such subintervals and $p_1, p_2, p_3, \ldots \ldots, p_k$ are the probabilities that the real root x^* lies in the $1^{st}, 2^{nd}, 3^{rd}, \ldots \ldots, k^{th}$ subinterval respectively. We assign,

p_i = prob{root lies in I_i}, i = 1, 2, 3, … …, k.

and later we show that $\sum_{i=1}^{k} p_i = 1$.

In iterative method of solution of a non linear equation, the functional value is decreased and approaches to zero if the sequence of iteration $\{x_n\}_{n=1}^{\infty}$ converges to the root x*. In this method we consider the decreasing rate(r_k) of the functional value of the function as follows:

$$r_k = \begin{cases} \frac{f(b_0) - f(a_0+\beta)}{b_0 - a_0 - \beta} \cdot \frac{b_0 - a_0}{k}, \; f(a_0) < 0 \;\&\; f(a_0+\beta) < 0 \\ \frac{f(a_0+\beta) - f(a_0)}{\beta} \cdot \frac{b_0 - a_0}{k}, \; f(a_0) < 0 \;\&\; f(a_0+\beta) > 0 \\ \frac{f(a_0) - f(a_0+\beta)}{\beta} \cdot \frac{b_0 - a_0}{k}, \; f(a_0) > 0 \;\&\; f(a_0+\beta) < 0 \\ \frac{f(a_0+\beta) - f(a_0)}{b_0 - a_0 - \beta} \cdot \frac{b_0 - a_0}{k}, \; f(a_0) > 0 \;\&\; f(a_0+\beta) > 0 \end{cases} \tag{1}$$

We divide the whole interval $[a_0, b_0]$ into k subintervals of equal lengths. Without loss of generality and for simplicity the value of k can be any one of 10, 10^2, 10^3, … ... i.e. we can take $k = 10^d$, *d* = 1, 2, 3, … …

We define n_d = the *d* digit integer whose digits are the first *d* digit of β after the decimal point.

For example,
let, β = 0.2777777777778, then for,

k = 10, (i.e. d = 1), n_d =2

k = 10^2, (i.e. d = 2), n_d =27

k = 10^3, (i.e. d = 3), n_d =277 and so on.

For β = 0.578085190331, then for

k = 10, (i.e. d = 1), n_d =5

k = 10^2, (i.e. d = 2), n_d =57

k = 10^3, (i.e. d = 3), n_d =578 and so on.

We assign the probabilities to each subinterval according to the following rules:

If $f(a_0 + \beta) < 0$

$$p_i = \begin{cases} 0, i \le n_d \\ \frac{1}{n_c - n_d} + (i - n_d - 1)r_k - (n_c - i)r_k, n_d < i \le n_c \\ 0, i > n_c \end{cases}$$

and if $f(a_0 + \beta) > 0$ (2)

$$p_i = \begin{cases} 0, i < n_c \\ \frac{1}{n_d - n_c} + (i - n_c - 1)r_k - (n_d - i)r_k, n_c \le i < n_d \\ 0, i \ge n_d \end{cases}$$

where n_c = celling $\left(\frac{|f(a_0)|}{r_k}\right)$.

From the above probability distribution it follows that the subinterval I_{n_c} have the maximum probability and we denote it P_{max} i.e.

$$P_{max} = P_{n_c} = \text{prob}\{\text{root lies in the subinterval } I_{n_c}\}$$

i.e. P_{max} occur in the subinterval I_{n_c} where $I_{n_c} = \left[a_0 + \frac{n_c - 1}{k}, a_0 + \frac{n_c}{k}\right]$

Now it is shown that the total probability $\sum_{i=1}^{k} p_i = 1$

$$\sum_{i=1}^{k} p_i = \sum_{i=1}^{n_d} p_i + \sum_{i=n_d+1}^{n_c} p_i + \sum_{i=n_c+1}^{k} p_i$$

$$= 0 + \sum_{i=n_d+1}^{n_c} p_i + 0$$

$$= \sum_{i=n_d+1}^{n_c} p_i$$

$$= \sum_{i=n_d+1}^{n_c} \left\{\frac{1}{n_c - n_d} + (i - n_d - 1)r_k - (n_c - 1)r_k\right\}$$

$$= \sum_{i=n_d+1}^{n_c} \frac{1}{n_c - n_d} + \sum_{i=n_d+1}^{n_c} (i - n_d - 1)r_k - \sum_{i=n_d+1}^{n_c} (n_c - i)r_k$$

$$= \frac{1}{n_c - n_d}\{n_c - (n_d + 1) + 1\} + 2r_k \sum_{i=n_d+1}^{n_c} i + r_k \sum_{i=n_d+1}^{n_c} (-1 - n_d - n_c)$$

$$= \frac{(n_c - n_d)}{(n_c - n_d)} + 2r_k \left\{\sum_{i=1}^{n_c} i - \sum_{i=1}^{n_d} i\right\} - r_k(1 + n_c + n_d)\{n_c - (n_d + 1) + 1\}$$

$$= 1 + 2r_k \left\{\frac{n_c(n_c + 1)}{2} - \frac{n_d(n_d + 1)}{2}\right\} - r_k(1 + n_c + n_d)(n_c - n_d)$$

$$= 1 + \frac{2r_k}{2}\{n_c(n_c + 1) - n_d(n_d + 1)\} - r_k(1 + n_c + n_d)(n_c - n_d)$$

$$= 1 + r_k\{n_c(n_c + 1) - n_d(n_d + 1)\} - (n_c - n_d)(1 + n_c + n_d)$$

$$= 1 + r_k\{n_c^2 + n_c - n_d^2 - n_d - n_c - n_c^2 - n_c n_d + n_d + n_c n_d + n_d^2\}$$

$$= 1 + r_k.0$$

$$= 1+0$$

$$= 1.$$

III. COMPUTATIONAL SCHEME

After assigning the probability as per rule (2) to the each subinterval, we have a subinterval I_{n_c} with maximum probability, where $I_{n_c} = \left[a_0 + \frac{n_c - 1}{k}, a_0 + \frac{n_c}{k}\right]$. Two cases may arise.

Case 1. The real root x^* lies in I_{n_c} then $x^* = a_0 + \frac{n_c - 1}{k}$, correct to *d* decimal places ($k = 10^d$) and we assign the next interval $[a_1, b_1] = \left[a_0 + \frac{n_c - 1}{k}, a_0 + \frac{n_c}{k}\right]$ which contain the root x^* such that $[a_1, b_1] \subset [a_0, b_0]$.

Case 2. The real root x^* does not lie in I_{n_c}. The computational scheme for finding the next interval $[a_1, b_1]$ which contain the root x^* is as follows:

Sub Case 2a). If $f(n_c) > 0$ then find an subinterval in which the real root x^* lies by checking the product of functional values at the each end points of each subintervals starting from I_{n_c} i.e. find an subinterval $[a_1, b_1] = \left[a_0 + \frac{n_c - 1}{k}, a_0 + \frac{n_c}{k}\right]$ such that $f\left(a_0 + \frac{n_c - i}{k}\right) f\left(a_0 + \frac{n_c - i - 1}{k}\right) < 0$ for a particular i (1, 2, 3, … …, n_c) so that the real root x^* lies in the interval.

Sub Case 2b). If $f(n_c) < 0$ then find an subinterval in which the real root x^* lies by checking the product of functional values at the each end points of each subintervals starting from I_{k-n_c} i.e. find an subinterval $[a_1, b_1] = \left[a_0 + \frac{n_c + i}{k}, a_0 + \frac{n_c + i + 1}{k}\right]$ such that $f\left(a_0 + \frac{n_c + i}{k}\right) f\left(a_0 + \frac{n_c + i + 1}{k}\right) < 0$ for a particular i = (0, 1, 2, 3, … …, k - n_c) so that the real root x^* lies in the interval.

So starting from $[a_0, b_0]$ we can find the next interval $[a_1, b_1]$ such that $[a_0, b_0] \supset [a_1, b_1]$. Processing in this way we can find out an interval $[a_n, b_n]$ which contain the real root x^* such that $[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \dots \dots \supset [a_n, b_n] \supset [a_{n+1}, b_{n+1}] \supset \dots \dots$

The computational cost of the proposed method:

I_{n_c} is the sub interval having the highest probability to contain the root x^*. To compute the intermediate I_{n_c} values, the above scheme takes nested n_c iteration. To compute the root x^* according to the desired accuracy, this scheme takes another n iterations. We have implemented this scheme in C and it is observed that the computational complexity of the algorithm is ***O(mn),*** where m = n_c.

IV. CONVERGENCE ANALYSIS

Two important factors which give the measure of successes of any iterative method for the solution of a nonlinear equation are order of convergence and rate of convergence. Let x_0 and x_n are the initial and n-th approximation of the root x^* of f(x) = 0. If there exists a number $p \geq 1$ and a constant $c \neq 0$ such that

$$\left|x^* - x_{n+1}\right| \leq c\left|x^* - x_n\right|^p$$

then p is called the order of convergence and c is the rate of convergence of the iterative method. Starting from $[a_0, b_0]$ we find the next interval $[a_1, b_1]$ such that

$[a_0, b_0] \supset [a_1, b_1]$ and $b_1 - a_1 = \frac{b_0 - a_0}{k}$

First approximation of the root is given by $x_0 = a_0 + \frac{n_c - 1}{k}$

In our proposed method we find out an interval $[a_n, b_n]$ which contain the real root x^* such that $[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \dots \dots \supset [a_n, b_n] \supset [a_{n+1}, b_{n+1}] \supset \dots ...$ and

$$b_2 - a_2 = \frac{b_1 - a_1}{k} = \frac{b_0 - a_0}{k^2},$$

$$b_3 - a_3 = \frac{b_2 - a_2}{k} = \frac{b_0 - a_0}{k^3},$$

… … …

… … …

$$b_n - a_n = \frac{b_{n-1} - a_{n-1}}{k} = \frac{b_0 - a_0}{k^n}.$$

From the above computational scheme 3 it is obvious that

$$\left|x^* - x_0\right| \le b_1 - a_1 = \frac{b_0 - a_0}{k}$$

$$\left|x^* - x_1\right| \le \frac{b_1 - a_1}{k} = \frac{b_0 - a_0}{k^2}$$

… … …

… … …

$$\left|x^* - x_n\right| \le \frac{b_n - a_n}{k} = \frac{b_0 - a_0}{k^{n+1}}$$

$$\left|x^* - x_{n+1}\right| \le \frac{b_{n+1} - a_{n+1}}{k} = \frac{b_0 - a_0}{k^{n+2}}$$

Therefore, we have

$$\frac{\left|x^* - x_{n+1}\right|}{\left|x^* - x_n\right|} \le \frac{b_0 - a_0}{k^{n+2}} \times \frac{k^{n+1}}{b_0 - a_0}$$

or, $$\frac{\left|x^* - x_{n+1}\right|}{\left|x^* - x_n\right|} \le \frac{1}{k}$$

or, $$\left|x^* - x_{n+1}\right| \le \frac{1}{k}\left|x^* - x_n\right|.$$

This shows that the proposed probabilistic method is linearly convergence. The rate of convergence of the proposed method is $\frac{1}{k}$. Bisection method and Regula-falsi method is also a linearly convergence and the rate of convergence of the Bisection method is $\frac{1}{2}$ and of the Regula-falsi method is $\le \frac{1}{2}$. So our proposed method is better than the Bisection method and Regula-falsi method for $k = 10^d$ (d = 1, 2, 3, … ...). But our proposed method is inferior to the Newton's method as the Newton's method is quadratic ally convergence.

V. EXPERIMENTAL RESULTS

Let us consider an equation f(x) = 3x – cosx – 1. The i^{th} subinterval in which the maximum probability of lying the real root x^* is n_c or P_{max} is computed. Let T_x^* be the subinterval where the real root x^* actually lie. The computed value of P_{max} and T_x^* are shown in TABLE I. It is seen that in all the cases except one or two, almost all of the computed values of P_{max} and T_x^* coincide. The procedure is implemented using programming language C.

TABLE I

COMPARISON BETWEEN PROBABILISTIC VALUE AND TRUE VALUE OF NUMBER OF INTERVAL

f(x)	**k**	**n**	**a_n**	**b_n**	**P_{max} or n_c**	**T_x^***
		0	0	1	6	7
		1	0.6	0.7	1	1
		2	0.60	0.61	8	8
		3	0.607	0.608	2	2
		4	0.6071	0.6072	1	1
		5	0.60710	0.60711	2	2
		6	0.607101	0.607102	7	7
		7	0.6071016	0.6071017	5	5
		8	0.60710164	0.60710165	9	9

f(x)	10	9	0.607101648	0.607101649	2	2
		10	0.6071016481	0.6071016482	1	1
		11	0.60710164810	0.60710164811	4	4
		12	0.607101648103	0.607101648104	2	2
		13	0.6071016481031	0.6071016481032	3	3
		14	0.60710164810312	0.60710164810313	3	3
		15	0.607101648103122	0.607101648103123	7	7
		16	0.6071016481031226	0.6071016481031227	4	4
	100	0	0	1	54	61
		1	0.60	0.61	72	72
		2	0.6071	0.6072	02	02
		3	0.607101	0.607102	65	65
		4	0.60710164	0.60710165	82	82
		5	0.6071016481	0.6071016482	04	04
		6	0.607101648103	0.607101648104	13	13
		7	0.60710164810312	0.60710164810313	27	27
		8	0.6071016481031226	0.6071016481031227	35	36
	1000	0	0	1	540	608
		1	0.607	0.608	102	102
		2	0.607101	0.607102	649	649
		3	0.607101648	0.607101649	104	104
		4	0.607101648103	0.607101648104	123	123
		5	0.607101648103122	0.607101648103123	636	636

We now present some numerical test results for various classical iterative schemes in TABLE II and TABLE III. Compared methods are Bisection method (BM), Regula–falsi method (RF), Newton's method (NM) and the proposed method (PM) with k = 10, k = 100, k = 1000. We use the following test functions and found the approximate zero (x^*) up to the 15 decimal places by using our probabilistic method (PM).

TABLE II

COMPARISON OF STANDARD CLASSICAL ITERATIVE SCHEMES IN NUMBER OF **IT**E**R**ATIONS (ITR)

f(x)	ITR					
	BM	RF	NM	PM k = 10	PM k = 100	PM k =1000
$f_1(x)$	51	26	5	15	8	5
$f_2(x)$	49	24	Not convergent	15	8	5
$f_3(x)$	53	129	7	15	8	5
$f_4(x)$	47	16	Failure	15	8	5

TABLE III

COMPARISON OF CLASSICAL ITERATIVE SCHEMES IN NUMBER OF FUNCTIONS EVALUATIONS (NFE)

f(x)	NFE					
	BM	RF	NM	PM k =10	PM k =100	PM k = 1000
$f_1(x)$	53	28	10	48	39	141
$f_2(x)$	51	26	-	51	71	468
$f_3(x)$	55	131	14	55	100	739
$f_4(x)$	49	18	-	47	28	37

$f_1(x) = x^3 - x - 11$, $[a, b] = [2, 3]$ $x^* = 2.373649822255812$
$f_2(x) = x - e^{\sin x} + 1$, $[a, b] = [1, 2]$ $x^* = 1.696812386809751$
$f_3(x) = 11x^{11} - 1$, $[a, b] = [0, 1]$ $x^* = 0.804133097503664$
$f_4(x) = xe^{-x} - 0.1$, $[a, b] = [0, 1]$ $x^* = 0.111832559158962$

Here we take the approximate solution, depending upon the precision (ε) of the computer. The stopping criteria $\left|x_{n+1} - x_n\right| < \varepsilon$ is used, and x^* is taken as the approximate root when the stopping criteria is satisfied. For all the numerical examples given in TABLE II and TABLE III the fixed stopping criteria, $\varepsilon = 1\times10^{-15}$ is used.

VI. Conclusion

In this method we find the approximate solution of a non linear equation by assigning a probability to each subinterval. This method is most suitable for a continuous smooth function. Although the convergence of the method is linear and the complexity of order *O(mn)*, but this idea of solution of nonlinear equation using probability is new and noble. In future course of study we shall give more emphasize to study the properties of β-value to identify the location of the root and to reduce the complexity of the method.

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