Local Approximation Estimates for a Certain Family of Summation-Integral Type Operators

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Abstract: In 1988, Felten obtained local and global approximation theorems for positive linear operators. In the last decade, Finta studied direct local and global approximation theorems for Baskakov type operators and Szasz-Mirakian type operators. In the present paper, we establish some local approximation estimates for a new family of beta operators, by using Ditzian-Totik modulus of smoothness.

Key Words and Phrases: Local approximation, Linear positive operators, Beta operators, K-functional, Ditzian-Totik modulus of smoothness.

MSC : 41A17, 41A36.

I. INTRODUCTION

In 1985, Upreti[10] studied approximation properties of beta operators[4, 5, 8]. Zhou [12] obtained direct and inverse theorems for these operators. Gupta and Dogru [6] obtained some direct results for beta operators. Recently, Kumar [9] studied direct results for Beta-Szasz operators in simultaneous approximation. Motivated by the work on beta operators, Gupta et al. [7] introduced a new family of beta operators to approximate lebesgue integrable functions on \([0, \infty)\) as

\[
B_n(f, x) = \frac{1}{(n + 1)} \sum_{v=1}^{\infty} b_{n,v}(x) \int_{0}^{\infty} b_{n,v}(t) f(t) \, dt, \quad x \in [0, \infty) \tag{1.1}
\]

where \(n \in \mathbb{N}\) (the set of natural numbers), \(b_{n,v}(t) = \frac{1}{\beta(v, n + 1)} t^{v-1} (1 + t)^{-n-v-1}\) and \(\beta(v, n + 1) = (v - 1)!n!(n + v)!\) the beta function.

Some basic properties of \(b_{n,v}(x)\) are as follows

(i). \(\int_{0}^{\infty} b_{n-v,v}(t) \, dt = 1\) \hspace{1cm} (1.2)

(ii). \(\int_{0}^{\infty} t b_{n,v}(t) \, dt = \frac{v}{n}\) \hspace{1cm} (1.3)

(iii). \(\int_{0}^{\infty} t^2 b_{n,v}(t) \, dt = \frac{v(v + 1)}{n(n - 1)}\) \hspace{1cm} (1.4)

(iv). \(\sum_{v=1}^{\infty} b_{n,v}(x) = (n + 1)\) \hspace{1cm} (1.5)

(v). \(\sum_{v=1}^{\infty} v b_{n,v}(x) = (n + 1)[1 + (n + 2)x]\) \hspace{1cm} (1.6)

(vi). \(\sum_{v=1}^{\infty} v^2 b_{n,v}(x) = (n + 1)[1 + 3(n + 2)x + (n + 2)(n + 3)x^2]\) \hspace{1cm} (1.7)

(vii). \(x(1 + x)b'_{n,v}(x) = [(v - 1) - (n + 2)x]b_{n,v}(x)\) \hspace{1cm} (1.8)
where \( n \in \mathbb{N} \) and \( x \in [0, \infty) \).

It can be easily verified that the operators (1.1) are linear positive operators and the order of approximation by these operators is at best \( O(n^{-\alpha}) \) as \( n \to \infty \), howsoever smooth the function may be.

Let \( L^1[0, \infty) \) be the class of functions \( g \) defined by

\[
L^1[0, \infty) = \{ g : g^{(n)} \in L^1[0, a] \text{ for every } a \in (0, \infty) \text{ and } g^{(n)}(t) \leq M (1 + t)^m \},
\]

where the constants \( M \) and \( m \) depend on \( g \), and \( L^p[a, b], 0 < p < \infty \) stands for the \( L^p \)-space.

It is obvious that \( L^p[0, \infty) \) is not contained in \( L^1[0, \infty) \).

Due to Ditzian and Totik\[1\], the modulus of smoothness of a function \( f \) is defined by

\[
\omega_{\phi}^2(f, t) = \sup_{0 < h < 2t} \left\| \Delta_{h\phi} f \right\|_p (1.9)
\]

where \( \phi^2(x) = x(1 + x) \) and

\[
\Delta_{h\phi} f(x) = \begin{cases} f(x - h) - 2 f(x) + f(x + h) & \text{if } [x - h, x + h] \subset [0, \infty) \\ 0 & \text{otherwise} \end{cases}
\]

Let \( \overline{W}^2_2(\phi, [0, \infty)) = \{ g \in L^2[0, \infty) : g' \in AC_{loc}[0, \infty) \text{ and } \phi^2 g'' \in L^2[0, \infty) \} \).

Following [14], it can be easily verified that the modulus of smoothness [1] defined by (1.9) is equivalent to the modified \( K \)-functional given by

\[
\overline{K}^2_2(f, t^2) = \inf \left\{ \| f - g \|_p + t^2 \| \phi^2 g'' \|_p : g \in \overline{W}^2_2(\phi, [0, \infty)) \right\}.
\]

The main object of the present paper is to establish some local approximation estimates [2, 3] for the operators (1.1) in terms of Ditzian-Totik modulus of second order.

### II. PRELIMINARY RESULTS

This section consists of some auxiliary results, which will be helpful in proving the main results of next section.

**Lemma 2.1.** For \( m, r \in \mathbb{N}^0 \) (the set of non-negative integers), let the function \( T_{r,m}(x) \) be defined as

\[
T_{r,m}(x) = \frac{1}{(n + r + 1)} \sum_{v=1}^{\infty} b_{n,v,r}(x) \int_0^\infty b_{n,r,v}(t) (t - x)^m dt.
\]

Then \( T_{r,0}(x) = 1 \), \( T_{r,1}(x) = \frac{(1 + r)(1 + 2x)}{(n - r)} \) \( (n > r) \), and for all \( n > m + r \), there holds the recurrence relation

\[
(n - m - r) T_{r,m+1}(x) = x(1 + x)[T'_{r,m}(x) + 2mT_{r,m-1}(x)] + (m + r + 1)(1 + 2x)T_{r,m}(x).
\]

Consequently, for each \( x \in [0, \infty) \), we have

\[
T_{r,m}(x) = O(n^{-(m+1)/2}),
\]

where \( [\alpha] \) denotes the integral part of \( \alpha \).

**Proof.** Using the definition of \( T_{r,m}(x) \) and basic properties of \( b_{n,v}(x) \), we obtain

\[
T_{r,0}(x) = 1 \quad \text{and} \quad T_{r,1}(x) = \frac{(1 + r)(1 + 2x)}{(n - r)}
\]

Now, we have

\[
x(1 + x)T'_{r,m}(x) = \frac{x(1 + x)}{(n + r + 1)} \sum_{v=1}^{\infty} b'_{n,v,r}(x) \int_0^\infty b_{n,r,v}(t) (t - x)^m dt - mx(1 + x)T_{r,m-1}(x)
\]

Therefore, using (1.8) we get

\[
x(1 + x)[T'_{r,m}(x) + mT_{r,m-1}(x)] = \frac{1}{(n + r + 1)} \sum_{v=1}^{\infty} [(v - 1) - (n + r + 2)x] b_{n,v,r}(x)
\]

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\[
\sum_{i=1}^{\infty} b_{n+r,i}(x) \int_0^\infty \left[ \left( v + r - 1 - (n - r + 2) t \right) + (n - r + 2) (t - x) + r(1+2x) \right] \nabla^n \cdot \nabla^r \cdot \nabla^m \cdot \left( b_{n, t} (t-x)^{m} \right) dt
\]

Similarly, if \( r > 0 \)

\[
\sum_{i=1}^{\infty} b_{n+r,i}^{(r)}(x) \int_0^\infty (1 + (n + r) t) b_{n-r, i+r}^{(r)}(t)(t-x)^{m} dt
\]

Again by Leibnitz theorem, we have

\[
\sum_{i=0}^{\infty} b_{n+r, i+r}(x) \int_0^\infty (1 + (n + r) t) b_{n-r, i+r}(t)(t-x)^{m} dt
\]

The other consequence easily follows from the above recurrence relation.

**Lemma 2.2.** For \( f \in \mathcal{L}^p_0[0, \infty) \cup \mathcal{L}^p_1[0, \infty) \), \( 1 \leq p \leq \infty \), \( n > r(m+1) \) and \( x \in [0, \infty) \), we have

\[
B_{n}^{(r)}(f, x) = \frac{\alpha(n, r)}{(n+1)} \sum_{i=1}^{\infty} b_{n+r,i}(x) \int_0^\infty b_{r-i, t}(t) f^{(i)}(t) dt
\]

where

\[
\alpha(n, r) = \frac{(n+r)! (N-r)!}{(n!)^2} = \prod_{j=1}^{r} \frac{(n+j)}{n - (j-1)}.
\]

**Proof.** By Leibnitz theorem, with the notation \( D = \frac{d}{dx} \), we have

\[
B_{n}^{(r)}(f, x) = \frac{1}{(n+1)} \sum_{i=1}^{\infty} b_{n+r,i}^{(r)}(x) \int_0^\infty b_{n-r,i}(t) f(t) dt
\]

Again by Leibnitz theorem, we have

\[
b_{n+r, i+r}(x) = \sum_{i=0}^{\infty} \frac{(n+r)!}{(n-r)!(v+r-1)!} \left( \int_0^\infty \left( \int_0^\infty \left( \int_0^\infty \cdots \int_0^\infty \frac{(r)(r+1)\cdots(r+i)}{i!} \right) b_{n+r,i}(t) f(t) dt \right) \right) dt
\]
Thus, from (2.2) and (2.3), we get

$$B^{(r)}_n (f, x) = \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} \frac{n!}{(n-i)!} \frac{r!}{(r-i)!} b_{n+r,i}(x)$$

(2.3)

Further integrating by parts \( r \) times, taking \( f(t) \) as first function, we get the required result (2.1).

Here, it is important to note that the operators defined in (2.1) by

$$B^{(r)}_n (f, x) = (B^r_n f)^{(r)}(x), \quad f \in L_p[0, \infty) \cup L^1[0, \infty)$$

are not positive. To make these operators positive, we introduce the operators

$$B_{n,r} f = D^r B^r_n f, \quad f \in L_p[0, \infty) \cup L^1[0, \infty)$$

where \( D \) and \( I \) stand for differentiation and integration operators respectively.

Thus, for all \( f \in L_p[0, \infty) \cup L^1[0, \infty) \) and \( n \geq n(1+m) \), the operators (2.1) can now be identified as

$$(B_{n,r} f)(x) = \sum_{i=0}^{\infty} (-1)^i \binom{n+r}{i} b_{n+i,i}(x) b_{n-r,i}(t) f(t) dt$$

Obviously, the operators \( B_{n,r} \) are positive and the estimation

$$\| (B^r_n f)^{(r)} - f^{(r)} \|_p, \quad f \in L_p[0, \infty)$$

is equivalent to

$$\| B_{n,r} f - f \|_p, \quad f \in L_p[0, \infty).$$

### III. LOCAL APPROXIMATION RESULTS

In this section, we establish direct local approximation theorems for the operators (1.1). Let \( C_B[0, \infty) \) be the space of all real valued continuous and bounded functions \( f \) on \([0, \infty)\) equipped with the norm \( \| f \| = \sup_{x \in [0, \infty)} |f(x)| \).

Also let

$$W^2_\infty = \{ g \in C_B[0, \infty) : g' \in C_B[0, \infty) \}.$$

Then, for \( \delta > 0 \), the \( K \)-functional are defined as

$$K_2(f, \delta) = \inf \{ \| f - g \| + \delta \| g' \| : g \in W^2_\infty \}.$$

If \( \omega(f, \delta) \) is the usual modulus of continuity of the function \( f \in C_B[0, \infty) \) defined by

$$\omega(f, \delta) = \sup_{h=0, \delta} \sup_{x \in [0, \infty)} |f(x+h) - f(x)|,$$

then \( \omega_2(f, \delta^{1/2}) \) the second order modulus of smoothness is defined as

$$\omega_2(f, \delta^{1/2}) = \sup_{h=0, \delta^{1/2}} \sup_{x \in [0, \infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

By Theorem 2.4 [11, p. 177], we can find a positive constant \( C_1 \) such that

$$K_2(f, \delta) \leq C_1 \omega_2(f, \delta^{1/2}) \quad \text{(3.1)}$$

Now we start to study the main results of this section, one of which stated as

**Theorem 3.1.** Let \( f \in C_B[0, \infty) \) and \( n \geq 2 \). Then for every \( x \in [0, \infty) \), there exists an absolute constant \( C_2 > 0 \) such that

$$|B_n(f,x) - f(x)| \leq C_2 \omega_2 \left( f, \sqrt{\frac{(x+1)(x+2)}{n}} \right) + \omega \left( f, \frac{1+2x}{n} \right) \quad \text{(3.2)}$$
Proof. To derive (3.2), we introduce a new operator

\[ \hat{B}_n : C_B(0, \infty) \to C_B(0, \infty) \]

defined by

\[ \hat{B}_n(f, x) = B_n(f, x) + f(x) - f\left(\frac{1 + (n + 2)x}{n}\right) \tag{3.3} \]

Using Lemma 2.1 for \( r=0 \), we get \( \hat{B}_n((t - x), x) = 0 \).

For \( t, x \in [0, \infty) \), by Taylor’s expansion of \( g \in W^2 \), we have

\[ g(t) = g(x) + (t - x)g'(x) + \int_x^t (t - y)g''(y) \, dy \]

Consequently, we have

\[ \hat{B}_n(g, x) - g(x) = \hat{B}_n\left(\int_x^t (t - y)g''(y) \, dy, x\right) \]
\[ = \hat{B}_n\left(\int_x^t (t - y)g''(y) \, dy, x\right) - \int_x^t \left(\frac{1 + (n + 2)x}{n} - y\right)g''(y) \, dy \tag{3.4} \]

Since

\[ \left| \int_x^t (t - y)g''(y) \, dy \right| \leq (t - x)^2 \| g'' \| \tag{3.5} \]

and

\[ \left| \int_x^t \left(\frac{1 + (n + 2)x}{n} - y\right)g''(y) \, dy \right| \leq \left(\frac{1 + (n + 2)x}{n} - x\right)^2 \| g'' \| \leq \left(\frac{1 + 2x}{n}\right)^2 \| g'' \| \leq \frac{4(x + 1)(x + 2)}{n^2} \| g'' \| \tag{3.6} \]

therefore, from (3.4), (3.5) and (3.6), we get

\[ \left| \hat{B}_n(g, x) - g(x) \right| \leq B_n((t - x)^2, x)\| g'' \| + \frac{4(x + 1)(x + 2)}{n^2} \| g'' \|. \]

Thus, using Lemma 2.1 for \( r=0 \), we obtain

\[ \left| \hat{B}_n(g, x) - g(x) \right| \leq \left[ \frac{2(n + 5)x^2 + 2(n + 5)x + 2}{n(n - 1)} + \frac{4(x + 1)(x + 2)}{n^2} \right] \| g'' \| \]
\[ \leq \left[ \frac{2(n + 5)}{(n - 1)} + \frac{4}{n} \right] \frac{(x + 1)(x + 2)}{n} \| g'' \| \]
\[ \leq \frac{16}{n} (x + 1)(x + 2) \| g'' \| \tag{3.7} \]

Further applying Lemma 2.1 for \( r=0 \), we get

\[ \left| B_n(f, x) \right| \leq \frac{1}{n + 1} \sum_{v=0}^\infty b_{n,v}(x)\int_0^\infty b_{n,v}(t) \left| f(t) \right| \, dt \leq \left\| f \right\| \]

which implies that \( B_n \) is a contraction i.e.

\[ \left\| B_n f \right\| \leq \left\| f \right\| \text{ for all } f \in C_B(0, \infty). \]
Now from (3.3), we get
\[
\| \hat{B}_n f \| \leq \| B_n f \| + 2 \| f \| \leq 3 \| f \| \quad \text{for all } f \in C_b[0, \infty) \tag{3.8}
\]
Hence, in view of (3.3), (3.7) and (3.8), we have
\[
\begin{align*}
\| B_n^*(f,x) - f(x) \| & \leq \| \hat{B}_n^*(f,x) - f(x) \| + \| (1 + (n + 2)x) f(x) - f(x) \| \\
& \leq \| \hat{B}_n^*(f) - g \| + \| g \| + \| \hat{B}_n^*(g,x) - g(x) \| + \| f(x) - f(1 + (n + 2)x) \| \\
& \leq 4 \| f - g \| + \frac{16}{n} (x + 1)(x + 2) \| g^* \| + \| f(x) - f(1 + (n + 2)x) \| \\
& \leq 16 \left[ \| f - g \| + \frac{(x + 1)(x + 2)}{n} \| g^* \| + \sup_{t, t' \in [1 + 2x/n, 2x]} \left| f(t) - f(t') \right| \right] \leq 16 \left( \| f - g \| + \frac{(x + 1)(x + 2)}{n} \| g^* \| + \omega \left( f, \frac{1 + 2x}{n} \right) \right) \tag{3.9}
\end{align*}
\]
Finally, taking the infimum on the right hand side of (3.9), over all \( g \in \mathcal{W}_c^2 \) and applying (3.1), we get the desired result (3.2).

Our next result of this section is the following

**Theorem 3.2.** Let \( n > r + 1 \geq 2 \) and \( f^{(i)} \in C_b[0, \infty) \), \( i = 0(1)r \). Then for every \( x \in [0, \infty) \), we have
\[
\begin{align*}
\| B_n^{(r)}(f,x) - f^{(r)}(x) \| & \leq (n + r + 1)! \left( \frac{1}{(n + 1)!} - 1 \right) \| f^{(r)} \|
+ \frac{(n + r + 1)!}{(n + 1)!} \left[ 1 + \sqrt{\frac{2(n + 1 + 2 + r(1 + r + 2)) x + 1 + (r + 1)(r + 2)}{n - r - 1}} \right]
\times \omega \left( f^{(r)}(x), (n - r)^{-1/2} \right) \tag{3.10}
\end{align*}
\]
**Proof.** Using the basic properties of \( b_{n,x}(x) \) and Lemma 2.2, we have
\[
\begin{align*}
B_n^{(r)}(f,x) - f^{(r)}(x) & \leq \frac{(n + r + 1)!}{(n + 1)!} \left[ \sum_{i=1}^{\infty} b_{n+r,x}^{(i)}(x) \int_0^{\infty} b_{n-r+1}^{(i)}(t) \left| f^{(r)}(t) - f^{(r)}(x) \right| dt \right]
+ \frac{(n + r + 1)!}{(n + 1)!} \left[ \frac{1}{(n + 1)!} - 1 \right] f^{(r)}(x)
\end{align*}
\]
Since for every \( \delta > 0 \), we have
\[
\left| f^{(r)}(t) - f^{(r)}(x) \right| \leq \omega \left( f^{(r)}, |t - x| \right) \leq \left[ 1 + \frac{|t - x|}{\delta} \right] \omega \left( f^{(r)}, \delta \right)
\]
therefore, we get
\[
\begin{align*}
\| B_n^{(r)}(f,x) - f^{(r)}(x) \| & \leq \frac{(n + r + 1)!}{(n + 1)!} \left[ \sum_{i=1}^{\infty} b_{n+r,x}^{(i)}(x) \int_0^{\infty} b_{n-r+1}^{(i)}(t) \left| f^{(r)}(t) - f^{(r)}(x) \right| dt \right]
+ \frac{(n + r + 1)!}{(n + 1)!} \left[ \frac{1}{(n + 1)!} - 1 \right] \| f^{(r)} \|
\end{align*}
\]
\[
\begin{align*}
\leq \frac{(n + r + 1)!}{(n + 1)!} \left[ \sum_{i=1}^{\infty} b_{n+r,x}^{(i)}(x) \int_0^{\infty} b_{n-r+1}^{(i)}(t) \left[ 1 + \delta^{-1} |t - x| \right] \omega \left( f^{(r)}, \delta \right) dt \right]
\end{align*}
\]
Applying Cauchy-Schwarz inequality for integration and then summation, we obtain
\[
\sum_{v=1}^{\infty} b_{n,r,v}(x) \int_0^\infty b_{n-r,v+r}(t) |t-x| dt \leq \sum_{v=1}^{\infty} b_{n+r,v}(x) \left[ \int_0^\infty b_{n-r,v+r}(t) dt \right]^{1/2} \left[ \int_0^\infty b_{n-r,v+r}(t) (t-x)^2 dt \right]^{1/2}
\]
\[
\leq \left( \sum_{v=1}^{\infty} b_{n+r,v}(x) \int_0^\infty b_{n-r,v+r}(t) (t-x)^2 dt \right)^{1/2}
\]
\[
\leq (n+r+1)^{1/2} \left( \sum_{v=1}^{\infty} b_{n+r,v}(x) \int_0^\infty b_{n-r,v+r}(t) (t-x)^2 dt \right)^{1/2}
\]

From straight computations, we get
\[
\int_0^\infty b_{n-r,v+r}(t) (t-x)^2 dt = \frac{(v+r+1)(v+r)}{(n-r)(n-r-1)} - 2x \frac{(v+r)}{(n-r)} + x^2
\]

Thus, we obtain
\[
\sum_{v=1}^{\infty} b_{n+r,v}(x) \int_0^\infty b_{n-r,v+r}(t) (t-x)^2 dt = \frac{(n+r+1)}{(n-r)(n-r-1)} \left\{ 2((n+1)+2(r+1)(r+2)) x(x+1)+(r+1)(r+2) \right\}
\]

Hence, collecting the estimates of (3.11), (3.12) and (3.13), we get
\[
\left| B_n^{(r)}(f,x) - f^{(r)}(x) \right| \leq \left[ \frac{(n+r+1)(n-r)!}{(n+1)!n!} - 1 \right] \left\| f^{(r)} \right\| + \frac{(n+r+1)(n-r)!}{(n+1)!n!} \left[ 1 + \delta^{-1} \right] \sqrt{\frac{2((n+1)+2(r+1)(r+2)) x(x+1)+(r+1)(r+2)}{(n-r)(n-r-1)}} \omega\left( f^{(r)}, \delta \right)
\]

Finally, choosing \( \delta = (n-r)^{-1/2} \), we get the required result (4.10).

This completes the proof of the Theorem 3.2.

REFERENCES


